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Numerical methods for calculation of stress and strain

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Consideration is given to the two main methods of numerical analysis which are currently used for solving problems in both small and large displacement elasticity, and in elasto-plastic problems. These are the finite element method, and the dynamic relaxation method. The former is of greater applicability than the latter, and is therefore treated in greater detail, so as to include methods for problems in solid mechanics which involve either one-, two-, or three-dimensional continua, as well as some problems in fluid mechanics such as flow through porous media.

Recent work in this field tends in the direction of more efficient utilization of the methods which have been developed during the past ten years. This means greater attention to computer storage organization and to more efficient solution routines. Essentially the choice here is between direct elimination and iteration methods, each of these two main categories having many varieties. The available methods are briefly discussed and some indication given of the more efficient computer methods.

1. Finite elements

The Rayleigh–Ritz procedure for approximate solutions in mechanics is well known, and consists of making sensible assumptions about the form of the unknowns in the problem. For example the deflexions are written in terms of Fourier series. The assumed forms satisfy boundary conditions, but they contain unknown parameters which are found by minimizing energy. Observe that the assumed form obtains for the whole structure, and the number of unknown parameters is small.

In the finite element procedure the structure is divided into a number of parts of finite size, in each of which assumptions are made about the form of deflexion and/or stress, in terms of a number of unknown parameters. These parameters are again found from variational principles. The finite element procedure is therefore similar to Rayleigh–Ritz, its advantages being chiefly in that polynomials can usually be assumed for deflexion or stress, for which the parameters are different for each element. Also each finite element can have different material properties. There are now many more unknown parameters to find and this produces computational problems (§ 3).

The structural engineer is familiar with such discretizing methods for frame structures, where beams are connected at nodes. This one-dimensional physical picture of finite element idealization is often extended, giving the impression of plate, shell or solid elements connected at node points only. This is an obviously erroneous picture. Much research has attempted to find assumed forms of displacement or stress which provide continuity, not only at node points, but at every point of an interelement boundary. It is true that the parameters are usually taken as displacement or stress at the nodes of the elements, and that these quantities are single-valued where elements meet at a node, but nodal values ultimately determine displacements throughout the element, and good elements are distinguished from bad by the single valuedness of displacements and their derivatives across common boundaries. If this continuity is violated, finite element methods are not a simple application of Rayleigh–Ritz (Oliveira 1971).

Historically, Courant (1943) presented an approximate solution to the St Venant torsion problem using a piecewise application of Rayleigh–Ritz, which involved the basic concepts of finite element methods. Significant contributions were also made by Prager & Synge (1947) with the introduction of the hypercircle method. From 1954 Argyris produced a series of papers on

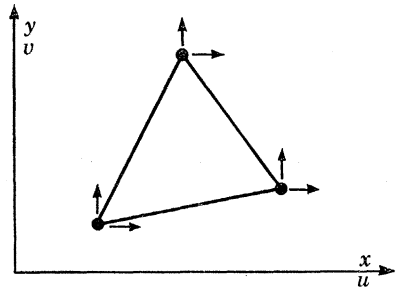

FIGURE 1. Plane stress–strain triangle with three nodes.

the matrix force method, but the beginning of the displacement approach is normally considered to be the paper by Turner, Clough, Martin & Topp (1956).

(a) *Finite element formulation: assumed displacements*

(i) *Plane stress–strain triangle with three nodes* (*figure* 1)

The formulation which has received most attention is based upon assumed displacement forms (Zienkiewicz 1971). In figure 1 a triangle has been isolated from two-dimensional space, and within it the displacements are

$$\left.\begin{aligned} u &= \beta_1+\beta_2 x+\beta_3 y, \\ \text{and}\quad v &= \beta_4+\beta_5 x+\beta_6 y, \end{aligned}\right\} \tag{1}$$

where $\beta_1 \ldots \beta_6$ are unknown coefficients, which may be determined in terms of the six nodal displacements by substituting nodal values of x and y in (1). The nodal displacements now become the unknowns. If q and β are the vectors of nodal displacements and unknown coefficients respectively, then $q = A\beta$, where A is a matrix of nodal values of x and y. The three components of strain are the derivatives of (1) so that, using ϵ to represent the strain vector, $\epsilon = B\beta$, where B is a matrix of functions of x and y in general. In this case,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y+\partial v/\partial x \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_6 \end{Bmatrix}. \tag{2}$$

If a relation between the stress vector, σ, and ϵ is postulated, such that $\sigma = R\epsilon$, then the stiffness matrix, k, for the element is

$$k = (A^{-1})^{\mathrm{T}} \int B^{\mathrm{T}} R B \,\mathrm{d}V A^{-1}, \tag{3}$$

where the integration is over the volume.

The stiffness matrix relates nodal displacements to nodal forces. It is square, symmetric and positive definite, and a general term k_{ij} is the force required corresponding to displacement i when displacement j is given unit value, and all other nodal displacements are zero. Thus,

$$f = kq. \tag{4}$$

A distinct advantage of finite elements is that (3) is independent of the shape of the element. But to achieve this generality it is necessary to use numerical integration (Irons 1966).

A stiffness matrix for each element can be obtained from (3), and when they are added together the nodal displacement vector for the whole structure, Q, is related to the vector of nodal forces, F, by

$$F = KQ. \tag{5}$$

The solution process consists first of obtaining Q from (5), and this is a most important computational aspect of the finite element method (§ 3). From Q, the q for each element is abstracted, from which the element strains and stresses are obtained, using the A, B and R matrices which have been held in the computer. Where two elements meet at a node, whether or not the stresses calculated from different elements are the same depends on the assumptions, such as (1), which have been made. Generally, they will not be the same, and the differences will depend upon applied loading, and whether the geometry changes abruptly. Unless such changes are severe it is satisfactory to average the calculable values at a node. This dilemma can be avoided by calculating strain and stress only at the centroid of the element.

The expression (3), obtained by the minimum potential energy principle, serves for plane stress and plane strain, only R being different for the two cases. But for plane strain of incompressible materials ($\nu = 0.5$), difficulties arise, and a different variational theorem is used, which is a special case of the Hellinger–Reissner principle (Herrmann 1965). This principle requires that stresses, as well as displacements, be assumed within the element, but as used by Herrmann the only independent variation of stress state permitted is that of average pressure.

(ii) *Ill-conditioning, compatibility and convergence*

The very flexibility of the finite element approach leads to difficulties. Thus, no real problems are encountered in dealing with anisotropic materials or with contiguous elements having dissimilar properties, but these situations can lead to ill-conditioning of stiffness matrices and consequent errors in solution.

A question to be asked of all finite element formulations concerns the conditions which must exist to ensure convergence as element size is reduced. The conclusion reached by Oliveira (1971) is that for the displacement formulation, convergence requires only that each of the displacements and their derivatives, up to the order used in the potential energy expression, are permitted to take up a constant value within the element. When the displacement field between elements is continuous as well, the elements are said to be compatible, and strain energy associated with this formulation is a true lower bound, with the displacements obtained converging monotonically as the elements are made smaller. In this sense, the linear displacement fields of (1) are compatible. It should not be thought that compatibility is a necessary condition for convergence. A valuable non-compatible element is the rectangular element for plate bending; this often produces faster convergence than similar compatible elements.

In discussing convergence, the problem of computer round-off errors should not be forgotten. General statements here are dangerous because much depends on the condition of the equations.

(iii) *Plane stress–strain triangle with six nodes*

Nodal displacements can be taken at points other than the apices. For example, if mid-side nodes are used

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{12} \end{Bmatrix}. \tag{6}$$

The A and B matrices are changed, but the stiffness matrix continues to be obtained from (3). It should be noted that complete polynomials have been used in (1) and (6). In plane elasticity the triangle is the 'natural' shape because the number of nodal displacements coincides with the number of terms in complete first- and second-order polynomials. If the chosen shape was a rectangle, with corner nodes, there would be eight nodal displacements. Various incomplete polynomials could be used, but none of these are satisfactory.

(iv) *Axisymmetric triangle*

A problem in plane elasticity is also provided when geometry and loading are symmetrical about the z-axis. The elements are then toroids of triangular cross-section with four components of strain, the integration in (3) is with respect to dr, dz and dθ, but the fields (1) or (6) are still used, with r and z replacing x and y. These displacement fields do not satisfy the differential equations of equilibrium in terms of the two displacement components (Allen, Pippard, Chitty & Severn 1956), but this is a feature of the displacement formulation which is not important because equilibrium is satisfied at all nodes.

When the structure remains geometrically axisymmetric, but the load does not, displacements and load may be expressed in Fourier series (Wilson 1965). Owing to the orthogonality of the harmonic functions, the non-axisymmetric problem is dealt with by solving n axisymmetric problems, where n is the number of harmonics required to represent the load.

(v) *Rectangular bending element* (*figure* 2)

For a plate element in bending, with transverse displacement w, compatibility between elements requires that consideration be given to the first derivatives of w as well as to w itself. At least, these derivatives must be continuous at node points, if not at all points on common boundaries.

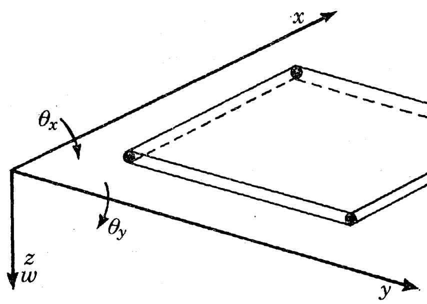

FIGURE 2. Rectangular bending element; assumed displacement approach.

For a rectangle with corner nodes this means that w, $\partial w/\partial x$ and $\partial w/\partial y$ must be chosen as nodal parameters, making twelve parameters in all. Without introducing unnecessary difficulties, the displacement formulation then requires that w be represented by a 12-term polynomial. The nearest complete polynomial is the cubic, with 10 terms, and to this are added a symmetrical pair of fourth-order terms, giving

$$w = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^3 + \beta_8 x^2 y + \beta_9 xy^2 + \beta_{10} y^3 + \beta_{11} x^3 y + \beta_{12} xy^3. \quad (7)$$

From this, the general formulation for the stiffness matrix, (3), can be used, taking curvatures as the 'strains' and stress resultants as the 'stresses'.

This element is not compatible, because although displacements and slopes along a common boundary are compatible, slopes normal to this boundary are not. However, this element has produced excellent results (Zienkiewicz 1971) in comparison with a compatible general quadrilateral element derived by De Veubeke (1968) by condensation from four triangular displacement fields.

Attempts at producing a triangular bending element directly from a single displacement field were not successful (Clough 1965). Such an element has nine nodal parameters and it is not possible to find a satisfactory polynomial containing just nine terms. A further difficulty is that the A matrix becomes singular when two sides of the triangle are parallel to the coordinate axes. A satisfactory bending triangle element, using the displacement formulation, was produced later by Zienkiewicz (1971), making use of area coordinates. Such triangles can be either compatible or non-compatible, both types giving satisfactory convergence.

(iv) *Isoparametric elements* (*figure* 3)

Computation associated with finite elements can be reduced if simple shapes are used because it is then possible to obtain the associated matrices in explicit form outside the computer. But the penalty suffered is that there are then restrictions on the accuracy with which shapes can be represented.

The principles to be used to remove these restrictions were proposed by Taig (1962), and later generalized by Ergatoudis, Irons & Zienkiewicz (1968), who gave to such elements the description 'isoparametric'.

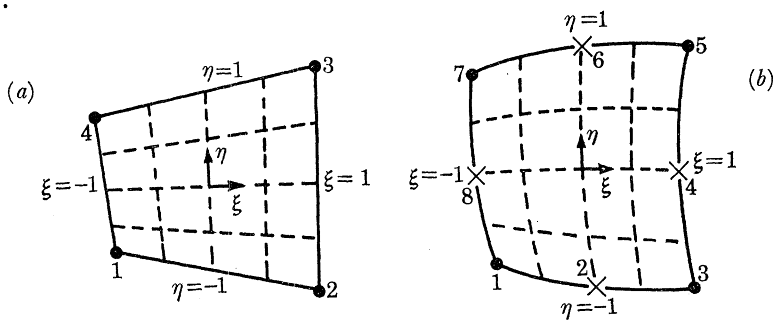

FIGURE 3. Two-dimensional isoparametric elements.

Referring to figure 3*a*, points 1, 2, 3, 4, in xy space can be joined by straight lines as shown to isolate a quadrilateral. But an alternative system of coordinates, $\xi\eta$, may be chosen such that ξ and η take the values ± 1 on the quadrilateral sides, as shown. Then,

$$\left.\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + \dots, \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + \dots, \end{aligned}\right\} \tag{8}$$

in which, by substitution of nodal coordinates in the two systems,

$$\left.\begin{aligned} 4N_1 &= (1-\xi)\,(1-\eta), \quad & 4N_2 &= (1+\xi)\,(1-\eta), \\ 4N_3 &= (1+\xi)\,(1+\eta), \quad & 4N_4 &= (1-\xi)\,(1+\eta). \end{aligned}\right\} \tag{9}$$

In general the $\xi\eta$ coordinates will be different for each element.

It is now assumed that displacement fields within the quadrilateral may be represented in the $\xi\eta$ coordinates by

$$\left.\begin{aligned} u(\xi,\eta) &= N_1 u_1 + N_2 u_2 + N_3 u_3 + \dots, \\ v(\xi,\eta) &= N_1 v_1 + N_2 v_2 + N_3 v_3 + \dots, \end{aligned}\right\} \tag{10}$$

where u and v are displacements in the x- and y-directions.

Using these assumed fields the stiffness matrix can be obtained from (3), but only after certain coordinate transformations. In terms of $\xi\eta$ the limits of integration in (3) are ± 1, and so it is convenient to work in these coordinates, but the strains must be referred to the xy axes, which are common to all elements. Thus,

$$\epsilon = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} = C \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ v_4 \end{Bmatrix}, \tag{11}$$

in which C is used to replace $A^{-1}B$. In (11), a typical term of the 3×8 matrix C is

$$C_i = \begin{bmatrix} \partial N_i/\partial x & 0 \\ 0 & \partial N_i/\partial y \\ \partial N_i/\partial y & \partial N_i/\partial x \end{bmatrix} \tag{12}$$

and, because $\xi\eta$ coordinates are used for the integration, these derivatives with respect to x and y must be transformed into derivatives with respect to ξ and η. This is a standard mathematical procedure using the Jacobian matrix, J. Also the area transformation is given by

$$\mathrm{d}x\,\mathrm{d}y = \det J\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{13}$$

If eight points are chosen in xy space (figure 3*b*), instead of four, the only new feature is that the N functions are now quadratic instead of linear.

The complete stiffness matrix calculation for the isoparametric element is carried out by numerical integration from a knowledge of the xy coordinates of the nodes, together with the material properties. Thus, no particular shape of element need be specified.

(vii) *Three-dimensional elements* (*figure* 4)

There is no difficulty in extending the isoparametric idea into three dimensions. All that is required is an additional equation in (8) and (10) for z and corresponding w. Figure 4 shows 20-node, 60-degree of freedom, element. A 32-node element, having two nodes within each edge, has also been used.

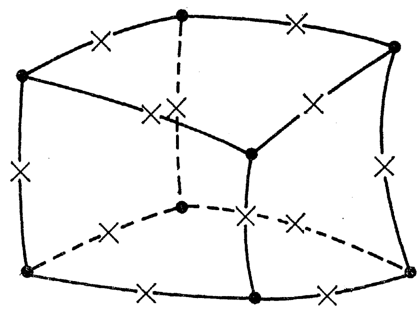

FIGURE 4. Three-dimensional, 20-node, isoparametric element.

With this kind of development, the finite element method faces the dilemma which confronted finite difference methods earlier; whether to have many simple elements, or few sophisticated ones? Adoption of the latter means a return towards Rayleigh–Ritz.

(*b*) *Other finite element formulations*

It is possible to obtain finite element properties other than by assuming displacement fields. Stress fields can be assumed (Veubeke 1965) within the element, or both displacement and stress fields can be assumed (Reissner 1950). A 'hybrid' approach has been presented by Pian which assumes stress fields within and on the boundaries and additionally assumes displacements on the boundaries only. This formulation can be used for both plane elasticity and bending (Severn & Taylor 1966), using triangular or quadrilateral elements. Its chief advantage over the displacement formulation is that the specification of boundary displacements allows compatible bending elements to be formed. It also allows the in-plane twist term to be introduced as a nodal parameter in the bending element. Some penalty is incurred here, because the specification of a single in-plane twist term at a node means that the angle there between the two sides of the element cannot change. Some workers have allowed two in-plane rotations at a node in order to overcome the difficulty.

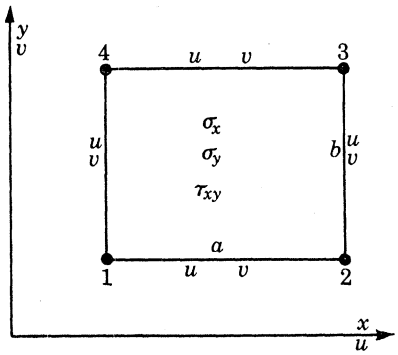

FIGURE 5. Plane stress–strain rectangle: hybrid approach, showing assumptions made.

The essential features of the hybrid approach can be illustrated by reference to figure 5, which shows a rectangular plane–stress element. Stress fields

$$\left.\begin{aligned}\sigma_x &= \alpha_1+\alpha_2 x+\alpha_3 y,\\ \sigma_y &= \alpha_4+\alpha_5 x+\alpha_6 y,\\ \tau_{xy} &= \alpha_7+\alpha_8 x+\alpha_9 y\end{aligned}\right\} \tag{14}$$

and

are assumed. These are not restricted to any particular order (but see Tong & Pian 1969), but for simplicity they are here taken to be linear. When these arbitrary stress fields are forced to satisfy equilibrium, certain relations are necessary between the α coefficients; here seven independent coefficients are retained and (14) can be written $\sigma = P\alpha$. If a stress–strain matrix, R, is assumed, the internal complementary energy can be calculated.

Considering the element boundary between nodes 1 and 2, u and v components of displacement are the nodal parameters. Between 1 and 2 a linear variation is assumed.

Thus,

$$\left.\begin{aligned}u &= u_1(1-x/a)+u_2(x/a),\\ v &= v_1(1-x/a)+v_2(x/a).\end{aligned}\right\} \tag{15}$$

Similarly, on other boundaries. The boundary forces are obtained from the boundary stresses, which are themselves obtained from (14) by giving x and y appropriate values, and these, with the displacements, allow the work done on boundaries to be calculated. The total work done within, and on the boundaries of, the element is now known in terms of the nodal displacements, and minimization with respect to these displacements gives the stiffness matrix.

This formulation may be applied to plane elements of different shapes, bending triangles and quadrilaterals of varying thickness (Dungar, Severn & Taylor 1967), and certain three-dimensional shapes. The necessary convergence criteria have been discussed by Tong & Pian

(1969) who produce several examples to show that this hybrid formulation produces faster convergence than other formulations.

(c) *Shell structures*

If a shell is thick, three-dimensional, isoparametric elements are used, but for normal shell geometry, the fact that the stiffnesses in the three directions are not of the same order can result in computational difficulties. Also this formulation ignores the fact that in shells plane sections remain plane after distortion. A special isoparametric formulation is therefore useful (Ahmad, Irons & Zienkiewicz 1970). Further simplifications can be made if the shell is axisymmetric.

For shells of moderate thickness success has been obtained using flat plate elements (Dungar & Severn 1968). An additional computational task is presented here because the stiffnesses of each element are related to its own axes, and transformation to common axes is necessary before stiffnesses can be added.

(d) *Nonlinear analysis*

A general discussion of nonlinear analysis has been given by Oden (1972). Here, only two applications will be noted. These are, the treatment of materials in which tension is not permitted (Zienkiewicz, Valliappan & King 1968), and elasto-plastic behaviour (Zienkiewicz, Valliappan & King 1969). The former is a special case of the latter.

Essentially, the approach adopted follows that of Allen & Southwell (1950). Restricting attention to plane elasticity, with triangular elements, elastic principal stresses are calculated at the centroid of each element. Where tensile stresses exist, they are made zero, and, to retain equilibrium, replaced by equivalent nodal forces. The calculation is then repeated with these nodal forces at the external loads. If tensile stresses now appear, the process is repeated.

When all elements have similar properties, this process is convergent, but difficulties arise if elements of high modulus are adjacent to elements of low modulus, and tension appears in the former. It is difficult to move the tension away from the high modulus elements.

In elasto-plastic analysis, an elastic analysis, with the full loads acting, is the first stage of the iteration. If the centroidal principal stresses are then calculated they may be applied to a chosen yield criterion to find the element which has most severely exceeded the criterion. The applied loads and resulting stresses may now be factored so that the most severely stressed element is about to yield, with all other elements still elastic. From this point in the calculation the residual applied load is applied in increments, of a size assessed largely by experience, until the full load is reached.

In the application of each load increment an elastic problem is solved to give an increment of strain, and corresponding stress. By virtue of the nonlinearity, this stress increment will be incorrect. But the stress which *is* consistent with the calculated strain can be obtained from the yield surface, and the difference between calculated and consistent stress can be interpreted as body force, which is fictitious, but which exists at this stage of the computation. This fictitious force is removed in a further elastic computation which produces additional strain and corresponding (elastic) stress. There will inevitably still be some difference between calculated and consistent stress, but this can be rectified as before. Only a few such iterations are usually necessary to bring the stress back to the yield surface. Further load increments are now applied, and on each application, elastic iterations are carried out to obtain a stress–strain situation confirming with the yield surface.

The advantage of this approach is that the stiffness matrix does not change during the iteration, and its inverse is obtained once only. What does change is the load vector.

In an alternative method for dealing with elasto-plasticity, a new stress–strain matrix is obtained at each iteration, and this means that the stiffness matrix must be recalculated and inverted. As an illustration of this, in the 'no-tension' problem, an element which indicates tension is made highly anisotropic for the next iteration, with a very low modulus in the tensile direction.

(e) *Concluding remarks*

In this review of finite elements it has not been possible to describe the evaluation of load vectors or mass matrices; nor to deal with thermal stress, creep or visco-elasticity. These present no new principles. In general, any problem which can be expressed in variational form can be solved by a finite element method.

2. Finite difference methods

In the pre-computer period extensive contributions were made by Southwell (1940, 1946, 1956), and Allen (1954). Their approach consisted of using the highest ordered equations which could be used to describe the problem, so that as few simultaneous algebraic equations as possible were presented for manual 'relaxation'. Difficulties were often experiences in satisfying the boundary conditions. A good illustration is given by the plate flexure problem. In terms of the deflexion, w, the governing equation is biharmonic. A simply supported edge is easily utilized, but on a free edge the moment and shear are zero, and this presents considerable practical difficulty.

Allen & Severn (1960) considered the flexural problem of slabs in terms of a pair of second-order equations and found that, although the number of algebraic equations had doubled, the imposition of boundary conditions was much simplified.

Attempts were made to implement the ideas of manual relaxation on the computer, but these ideas had become too sophisticated to make the transfer successful.

Dynamic relaxation

'Relaxation' is used in the sense used by Southwell (1940). The structure is constrained before load is applied. These artificial constraints are then systematically relaxed to zero, allowing the structure to adopt an equilibrium position. 'Dynamic' refers to the fact that the equations solved contain inertia and viscous damping terms. The static equilibrium condition is obtained by iteration from an initial state, with damping as near critical as possible. The problems associated with this method are concerned with finding the damping value and the maximum time-step which can be used while still giving convergence. Because there is no interest in the route by which the solution is obtained, there is no need to use the *actual* density. Fictitious densities can be used, with the aim of producing a maximum rate of convergence (Cassell 1970).

The equations to be solved by dynamic relaxation need not arise from a finite difference treatment of differential equations. They can arise from a finite element formulation (Taylor 1965; Lynch, Kelsey & Saxe 1968). However, it is with finite differences in mind that dynamic relaxation is considered here, and the principles can be seen from an example (Otter 1965).

A bar of uniform cross-section is fixed at one end and a constant stress applied at the other. This stress exists at all cross-sections and the problem is trivial, but the formulation is typical. Using p and u for stress and displacement, the stress–strain and equilibrium equations are

$$p = E\,\partial u/\partial x \quad \text{and} \quad \partial p/\partial x = 0. \tag{16}$$

Statical equilibrium is now converted into dynamical equilibrium by adding inertia and damping. Because inertia contains the second derivative of displacement it is convenient to introduce velocity $\dot{u}$ as a variable. Thus (16) becomes

$$\frac{\partial p}{\partial t} - E\frac{\partial \dot{u}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p}{\partial x} - \rho\frac{\partial \dot{u}}{\partial t} - k\dot{u} = 0; \tag{17}$$

a pair of first-order equations with the velocity and stress as dependent variables. In finite difference form (17) give

$$p_i = p_j + E\frac{\Delta t}{\Delta x}(\dot{u}_i - \dot{u}_j) \tag{18}$$

and

$$u_i = \frac{1}{1+K}\left\{(1-K)\,\dot{u}_j + \frac{\Delta t}{\rho \Delta x}(p_I - p_J)\right\}, \tag{19}$$

where $K = \frac{1}{2}k\Delta t$. Lower case suffices are used in the time direction; upper case suffices in the x-direction. An iterative scheme based upon (18) and (19) presents no difficulties.

The boundary conditions here are a constant stress condition at one end and a constant velocity at the other. It is therefore convenient to use interlacing nets (Gilles 1948). Thus stress and velocity values are not calculated at the same points of the bar.

Because stresses are conveniently used in the dynamic relaxation method, nonlinear stress–strain behaviour is easily dealt with. In fact, one of its earliest applications (Day 1965) was to a thick steel cylinder under internal pressure, using an elastic – perfectly plastic stress–strain curve. Earlier, Otter & Day (1960) had studied tidal flow. More recently (Otter, Cassell & Hobbs 1966), the arch dam problem has been solved, and Rushton (1968 *a*, *b*) has produced solutions for large deflexion of plates, while Cassell (1970) has studied shells.

Two difficulties arise in dynamic relaxation. First, if it is to retain its advantage of dealing easily with mixed boundary conditions, the surface of the structure must be expressible in orthogonal coordinates. Secondly, the optimum time step and damping parameter are not known initially.

(i) *Optimum time step*

If Δt is too small, computer time is wasted; if it is too large the iteration will diverge. This numerical stability problem was considered by Forsythe & Wasow (1960). Physically, the progress of the calculations considered as a wave must outrun the wave corresponding to the physical problem.

As an example, for the problem of large deflexion of a plate, with a square mesh of side Δx, stability considerations linked with the extensional wave require that

$$\Delta t \leqslant \left\{\frac{2\rho_u(1+\nu)\,(1-2\nu)}{1-\nu}\right\}^{\frac{1}{2}}, \tag{20}$$

and with the flexural wave that

$$\Delta t \leqslant 0.25\rho_w^{\frac{1}{2}}(\Delta x)^2. \tag{21}$$

It will generally be found that (21) is governing.

In many problems it is expedient to have a graded mesh. This seems to imply that the largest allowable time interval would be given by the smallest mesh size, and this would be wasteful. Cassell (1970) has shown that variable fictitious densities may be used.

(iii) *Damping factor*

For a single degree of freedom system, critical damping can be obtained from the natural frequency. In dynamic relaxation there are many degrees of freedom, but Rushton (1968 *b*) concludes that for beams and plates, if the damping is taken as critical damping for the fundamental mode, contributions from higher modes are damped out within two cycles of the fundamental mode. For an arch dam, Otter *et al.* (1966) found the same to be true, taking the fundamental frequency of vibration from the experimental work of Taylor (1965).

In general, the fundamental frequency of the structure is not known, and if fictitious densities are used it is dangerous to estimate it intuitively, but an approximation may be obtained by allowing the iteration to start with zero damping, and following the oscillation produced. This process is more useful if total kinetic energy is plotted against the number of iterations.

3. Computational aspects

In the finite element method large systems of algebraic equations must be solved. These equations are linear, symmetric, sparse and banded, positive definite, and usually diagonally dominant. Efficient solution requires that note be taken of these characteristics, and assuming that care has been taken to produce minimum band-width by proper nodal ordering, the choice of method lies between direct elimination and iteration.

A summary and appraisal of the various methods has recently been given by Tuff (1971), and it appears that direct methods are generally preferable. Of the direct elimination methods, the fact that the matrix of coefficients is positive definite makes the Choleski triangulation process very attractive.

Iterative procedures are less demanding in storage requirements, but more demanding in computer time unless acceleration factors are used. For matrices possessing Young's (1954) property *A*, the optimum acceleration factor can be found, but the matrices which are produced by the finite element method do not possess this special property and recourse must be had to trial and error.

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